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Trilocal Structures. II. Expansion

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A procedure is described by which a set of expansion functions is generated for the trilocal wave function. The first 49 functions in the set are listed, along with matrix elements of the Hamiltonian that are generated through this same procedure.

1. INTRODUCTION

In the previous article in this set (Clapp et al., 1978), which will be referred to as I, the secular equation for the trilocal structure was derived and solved. The auxiliary parameters λ and μ were found to satisfy the equations (I.4.5a) and (I.4.5b), in order that the structure should move relativistically as a particle with the rest mass m, but the secular equation did not itself place any requirements on m, except that m^2 should lie in the range between zero and nine, inclusive of these limits.

The curve which showed $\cos(3\mu)$ as a function of m contained horizontal tangents at $m = 0$ and $m = (9/33)^{1/2}$. The significance of these two special mass values will emerge later in the analysis.

The present article will be concerned with the expansion of the trilocal wave function in terms of an infinite set of orthogonal functions, each of which satisfies the phase–space boundary condition given previously in (I.2.11).

2. REST SYSTEM EXPANSION

The trilocal wave function can be viewed as lying in four tiers, corresponding to the τ-spin projections with $M_\tau = +\frac{3}{2}, +\frac{1}{2}, -\frac{1}{2}$, and $-\frac{3}{2}$. Within any one of these four tiers the centroid-time wave equation (I.2.8a) couples together the orthogonal functions in the expansion set. At a later stage a new

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operator will be introduced which will couple together functions on different tiers.

Initially, we will restrict the solutions to the rest system, with

$$\mathbf{k} = 0 \tag{2.1}$$

We will also restrict our attention to the leading portion of the wave function, considered as a power-series expansion in the two relative-time variables t_r and t_ρ, which were defined in (I.2.2). That is, we will consider the part of the full wave function that remains when we set $\mathbf{k}$, t_r, and t_ρ all equal to zero.

Antisymmetry will be required. On the two outer tiers, $M_\tau = \pm\frac{3}{2}$, antisymmetry must come via space and σ spin; with three quanta in the structure there can be no function that is nonzero at the center of the structure where $\mathbf{r} = \boldsymbol{\rho} = 0$.

On the two inner tiers, $M_\tau = \pm\frac{1}{2}$, we can use both τ spin and σ spin for antisymmetry, and include in the expansion a function which does not vanish for $\mathbf{r} = \boldsymbol{\rho} = 0$. For the upper of these two inner tiers, we can define this innermost function by

$$\psi_1^{+1/2} = N_0 h_0(\kappa\mathscr{R})|{}^b\Gamma^{+1/2\ 2c}(1) - {}^c\Gamma^{+1/2\ 2b}(1)] \tag{2.2}$$

where

$$N_0 = \kappa^{3/2}[2\cdot(6^{1/2})] \tag{2.3}$$

is a common normalizing factor for each function, which gives each function the dimension of $(\text{length})^{-3/2}$, so that its square is a volume density.

The hyperspherical radial variable $\mathscr{R}$ is defined by

$$\mathscr{R}^2 = 2r^2 + 3\rho^2/2 = (\mathbf{r}_1 - \mathbf{r}_2)^2 + (\mathbf{r}_2 - \mathbf{r}_3)^2 + (\mathbf{r}_3 - \mathbf{r}_1)^2 \tag{2.4}$$

The function $h_0(z)$ is a hyperspherical Bessel function, analogous to the familiar spherical Bessel function $j_0(z)$. As in Clapp (1978), the set of functions $h_n(z)$ will be defined in such a way that the composite functions each satisfy the phase–space boundary condition (I.2.11). The functions $h_n(z)$ are defined by

$$h_n(z) = 8z^{-n-2}J_{n+2}(z) \tag{2.5}$$

in terms of the ordinary Bessel functions. These h_n satisfy the recursion relation

$$h_n(\kappa\mathscr{R}) = \frac{1}{2n+4}[h_{n-1}(\kappa\mathscr{R}) + \kappa^2\mathscr{R}^2 h_{n+1}(\kappa\mathscr{R})] \tag{2.6}$$

Differentiation of the h_n gives

$$\frac{d}{d\mathscr{R}} h_n(\kappa\mathscr{R}) = -\kappa^2\mathscr{R} h_{n+1}(\kappa\mathscr{R}) \tag{2.7}$$

The σ-spin functions ${}^{2c}(1)$ and ${}^{2b}(1)$ are the functions defined in Clapp (1961) as ${}^{2c}(1)_{1/2}$ and ${}^{2b}(1)_{1/2}$. These can be represented as 8-component column vectors. For the top component, the three σ spins are all positive. The next component has σ_{1z} negative, σ_{2z} and σ_{3z} positive. The pattern can be represented by the format

$$\begin{array}{ccc} 1 & 2 & 3 \\ \hline + & + & + \\ \hline - & + & + \\ + & - & + \\ + & + & - \\ \hline + & - & - \\ - & + & - \\ - & - & + \\ \hline - & - & - \\ \hline \end{array}^{\sigma} \tag{2.8}$$

which was given earlier as (2) of Clapp (1961).

The ${}^2S$ σ-spin function ${}^{2b}(1)$ has the two components

$$ {}^{2b}(1)^{+1/2} = \begin{array}{c} \hline 0 \\ \hline 0 \\ 1 \\ -1 \\ \hline 0 \\ 0 \\ 0 \\ \hline 0 \\ \hline \end{array}^{\sigma} \qquad {}^{2b}(1)^{-1/2} = \begin{array}{c} \hline 0 \\ \hline 0 \\ 0 \\ 0 \\ \hline 0 \\ -1 \\ 1 \\ \hline 0 \\ \hline \end{array}^{\sigma} \tag{2.9}$$

The other ^2S σ-spin function has the two components

$$
{}^{2c}(1)^{+1/2} = \begin{array}{c} \hline 0 \\ \hline 2 \\ -1 \\ -1 \\ \hline 0 \\ 0 \\ 0 \\ \hline 0 \\ \hline \end{array}^{\sigma} \qquad {}^{2c}(1)^{-1/2} = \begin{array}{c} \hline 0 \\ \hline 0 \\ 0 \\ 0 \\ \hline -2 \\ 1 \\ 1 \\ \hline 0 \\ \hline \end{array}^{\sigma} \tag{2.10}
$$

The function $\psi_{\mathrm{I}}^{+1/2}$ also involves two τ-spin functions. The τ-spin function ${}^{b}\Gamma^{+1/2}$ is analogous to ${}^{2b}(1)^{+1/2}$ in (2.9), but with the superscript σ changed to τ. The τ-spin function ${}^{c}\Gamma^{+1/2}$ is similarly analogous to ${}^{2c}(1)^{+1/2}$ in (2.10), again with the superscript changed from σ to τ.

We can also use the cyclic forms for the τ-spin and σ-spin functions. We can write

$$
{}^{2+}(1) = (1/2)\,{}^{2c}(1) \quad [i(3)^{1/2}/2]\,{}^{2b}(1) \tag{2.11a}
$$

$$
{}^{2-}(1) = (1/2)\,{}^{2c}(1) - [i(3)^{1/2}/2]\,{}^{2b}(1) \tag{2.11b}
$$

This gives us

$$
{}^{2+}(1)^{+1/2} = \begin{array}{c} \hline 0 \\ \hline 1 \\ \omega \\ \omega^2 \\ \hline 0 \\ 0 \\ 0 \\ \hline 0 \\ \hline \end{array}^{\sigma} \qquad {}^{2+}(1)^{-1/2} = \begin{array}{c} \hline 0 \\ \hline 0 \\ 0 \\ 0 \\ \hline -1 \\ -\omega \\ -\omega^2 \\ \hline 0 \\ \hline \end{array}^{\sigma} \tag{2.12}
$$

and

$$
{}^{2-}(1)^{+1/2} = \begin{array}{c} \hline 0 \\ \hline 1 \\ \omega^2 \\ \omega \\ \hline 0 \\ 0 \\ 0 \\ \hline 0 \\ \hline \end{array}^{\sigma} \qquad {}^{2-}(1)^{-1/2} = \begin{array}{c} \hline 0 \\ \hline 0 \\ 0 \\ 0 \\ \hline -1 \\ -\omega^2 \\ -\omega \\ \hline 0 \\ \hline \end{array}^{\sigma} \tag{2.13}
$$

These functions correspond to (28–31) of Clapp (1961).

As for the cyclic forms of the τ-spin functions, we will want to use a different phasing, in order to simplify the formulas that will later be used for families of expansion functions. We will define

$$
{}^{+}\Gamma^{+1/2} = \begin{array}{c} \hline 0 \\ \hline 1 \\ \omega \\ \omega^2 \\ \hline 0 \\ 0 \\ 0 \\ \hline 0 \\ \hline \end{array}^{\tau} \qquad {}^{+}\Gamma^{-1/2} = \begin{array}{c} \hline 0 \\ \hline 0 \\ 0 \\ 0 \\ \hline 1 \\ \omega \\ \omega^2 \\ \hline 0 \\ \hline \end{array}^{\tau} \tag{2.14}
$$

and

$$
{}^{-}\Gamma^{+1/2} = \begin{array}{c} \hline 0 \\ \hline 1 \\ \omega^{2} \\ \omega \\ \hline 0 \\ 0 \\ 0 \\ \hline 0 \\ \hline \end{array}^{\tau} \qquad {}^{-}\Gamma^{-1/2} = \begin{array}{c} \hline 0 \\ \hline 0 \\ 0 \\ 0 \\ \hline 1 \\ \omega^{2} \\ \omega \\ \hline 0 \\ \hline \end{array}^{\tau} \tag{2.15}
$$

We will also define

$$
{}^{s}\Gamma^{+3/2} = \begin{array}{c} \hline (3)^{1/2} \\ \hline 0 \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ 0 \\ \hline 0 \\ \hline \end{array}^{\tau} \qquad {}^{s}\Gamma^{+1/2} = \begin{array}{c} \hline 0 \\ \hline 1 \\ 1 \\ 1 \\ \hline 0 \\ 0 \\ 0 \\ \hline 0 \\ \hline \end{array}^{\tau} \tag{2.16a}
$$

$$
{}^{s}\Gamma^{-1/2} = \begin{array}{c} \underline{}_{\tau} \\ 0 \\ \hline 0 \\ 0 \\ 0 \\ \hline 1 \\ 1 \\ 1 \\ \hline 0 \\ \hline \end{array} \qquad {}^{s}\Gamma^{-3/2} = \begin{array}{c} \underline{}_{\tau} \\ 0 \\ \hline 0 \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ 0 \\ \hline (3)^{1/2} \\ \hline \end{array} \qquad (2.16b)
$$

With these definitions, we can rewrite $\psi_1^{+1/2}$ in the form

$$
\psi_1^{+1/2} = N_0[2i/(3)^{1/2}]h_0(\kappa\mathscr{R})[{}^{-}\Gamma^{+1/2\;2+}(1) - {}^{+}\Gamma^{+1/2\;2-}(1)] \qquad (2.17)
$$

This function, as noted above, belongs to the upper of the two inner tiers. There will be a similar function, which we can write as $\psi_1^{-1/2}$, belonging to the lower of the two inner tiers, where we have $M_\tau = -\frac{1}{2}$. In a full trilocal structure, these will have separate coefficients, which we can denote by $C_1^{+1/2}$ and $C_1^{-1/2}$.

We will want to define the two operators

$$
\tau^{+} = (\tau_{1\zeta}\tau_{2\zeta}\tau_{3\zeta})(\tau_{1\zeta} + \omega\tau_{2\zeta} + \omega^2\tau_{3\zeta})/2 \qquad (2.18a)
$$

$$
\tau^{-} = (\tau_{1\zeta}\tau_{2\zeta}\tau_{3\zeta})(\tau_{1\zeta} + \omega^2\tau_{2\zeta} + \omega\tau_{3\zeta})/2 \qquad (2.18b)
$$

These operators act on the τ-spin functions (2.14–2.16). They annihilate the outer-tier functions ${}^{s}\Gamma^{\pm 3/2}$. However, acting on the inner-tier functions, they permute them cyclically according to the scheme

$$
\tau^{+}\,{}^{s}\Gamma^{\pm 1/2} = {}^{+}\Gamma^{\pm 1/2} \qquad \tau^{-}\,{}^{s}\Gamma^{\pm 1/2} = {}^{-}\Gamma^{\pm 1/2} \qquad (2.19a)
$$

$$
\tau^{+}\,{}^{+}\Gamma^{\pm 1/2} = {}^{-}\Gamma^{\pm 1/2} \qquad \tau^{-}\,{}^{+}\Gamma^{\pm 1/2} = {}^{s}\Gamma^{\pm 1/2} \qquad (2.19b)
$$

$$
\tau^{+}\,{}^{-}\Gamma^{\pm 1/2} = {}^{s}\Gamma^{\pm 1/2} \qquad \tau^{-}\,{}^{-}\Gamma^{\pm 1/2} = {}^{+}\Gamma^{\pm 1/2} \qquad (2.19c)
$$

Thus, within these inner tiers, the two operators (2.18) satisfy the operator relationships

$$
\tau^{+}\tau^{-} = 1 = (\tau^{+})^3 = (\tau^{-})^3 \qquad (\tau^{+})^2 = \tau^{-} \qquad (\tau^{-})^2 = \tau^{+} \qquad (2.20)
$$

which are the same identities satisfied by the complex numbers ω and ω^2, given previously in (I.2.5).

Along with the operators (2.18), we will want to use the σ-spin operators defined by

$$\boldsymbol{\sigma}^s = \boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2 + \boldsymbol{\sigma}_3 \qquad \boldsymbol{\sigma}^+ = \boldsymbol{\sigma}_1 + \omega\boldsymbol{\sigma}_2 + \omega^2\boldsymbol{\sigma}_3$$
$$\boldsymbol{\sigma}^- = \boldsymbol{\sigma}_1 + \omega^2\boldsymbol{\sigma}_2 + \omega\boldsymbol{\sigma}_3 \tag{2.21}$$

We can then write the centroid wave equation, as specialized to the $M_\tau = +\frac{1}{2}$ tier, in the form

$$w\Phi^{+1/2} = H^{+1/2}\Phi^{+1/2} \tag{2.22a}$$

where

$$H^{+1/2} = H_k^{+1/2} + H_r^{+1/2} \tag{2.22b}$$

$$H_k^{+1/2} = (1/9)(\boldsymbol{\sigma}^s\cdot\mathbf{k} - \tau^+\boldsymbol{\sigma}^-\cdot\mathbf{k} - \tau^-\boldsymbol{\sigma}^+\cdot\mathbf{k}) \tag{2.22c}$$

$$H_r^{+1/2} = (1/9i\kappa)(-\tau^+\boldsymbol{\sigma}^s\cdot\boldsymbol{\nabla}^- - \tau^-\boldsymbol{\sigma}^s\cdot\boldsymbol{\nabla}^+ + \boldsymbol{\sigma}^+\cdot\boldsymbol{\nabla}^- + \boldsymbol{\sigma}^-\cdot\boldsymbol{\nabla}^+ - \tau^+\boldsymbol{\sigma}^+\cdot\boldsymbol{\nabla}^+ - \tau^-\boldsymbol{\sigma}^-\cdot\boldsymbol{\nabla}^-) \tag{2.22d}$$

In the rest system, specified by (2.1), $H_k^{+1/2}$ in (2.22c) will vanish, leaving only the relative part of the Hamiltonian, $H_r^{+1/2}$ in (2.22d). When this operator acts on the initial function $\psi_1^{+1/2}$, as given in (2.17), two other functions are generated. Operation on each of these regenerates (2.17), together with other new functions in the expansion of $\Phi^{+1/2}$, which is the portion of the wave function lying on this upper-middle tier.

3. NORMALIZATION

In the rest system, the energy w reduces to the rest mass m, and the wave equation (2.22a) simplifies to

$$m\Phi^{+1/2} = H_r^{+1/2}\Phi^{+1/2} \tag{3.1}$$

When the operator in (3.1) is applied to the initial function (2.17), and to the succession of functions that are generated, a matrix version of this operator is constructed, with this set of functions as the basis system.

The functions are first generated with arbitrary normalization factors. The matrix version of the relative Hamiltonian operator is then not a symmetrical matrix. However, the matrix can be forced into a symmetrical form through the introduction of normalization factors to accompany the separate functions. This normalization is a relative one that takes the normalization of the initial function (2.17) as a given quantity.

We should note here that the usual picture of a normalized wave function does not apply, since the initial function (2.17), along with each of the functions generated by the operator $H_r^{+1/2}$, is not square-integrable. In this theory we are working with waves that converge to a center and then diverge to infinity. These spherical waves are no more square-integrable than are the plane waves which are utilized in quantum-mechanical scattering calculations. In analogy with the artificial large box that is introduced to provide a normalization for the plane waves, we might introduce an artificial large sphere, with a radius of the Hubble distance or some such length characterizing the observable universe. However, for the present analysis we are concerned only with relative normalization, not with absolute normalization, so we can postpone consideration of the latter, and deal only with the former.

When we adjust normalization factors to make the relative Hamiltonian in (3.1) a symmetrical matrix, we find that there are usually more conditions to be satisfied than there are adjustable factors. The extra conditions provide welcome checks on the algebra.

4. LIST OF FUNCTIONS

The first 49 of the inner-tier trilocal basis functions are listed in Appendix A. Sixteen of these, the ones that incorporate the τ-spin function ${}^s\Gamma$, will also appear in the two outer tiers, though with separately adjustable coefficients.

Sixteen σ-spin rotational functions were introduced in Clapp (1961), where they were given the mnemonics

$$\begin{array}{cccccccc} {}^{2b}(1) & {}^{2c}(1) & {}^{2b}(\mathbf{r}) & {}^{2c}(\mathbf{r}) & {}^{2b}(\boldsymbol{\rho}) & {}^{2c}(\boldsymbol{\rho}) & {}^{4}(\mathbf{r}) & {}^{4}(\boldsymbol{\rho}) \\ \end{array}$$
$$\begin{array}{ccccc} {}^{2b}(i\mathbf{r}\times\boldsymbol{\rho}) & {}^{2c}(i\mathbf{r}\times\boldsymbol{\rho}) & {}^{4}(i\mathbf{r}\times\boldsymbol{\rho}) & {}^{4}(\mathbf{rr}) & {}^{4}(\boldsymbol{\rho\rho}) \end{array}$$
$$\begin{array}{ccc} {}^{4}(\mathbf{r}\boldsymbol{\rho}+\boldsymbol{\rho}\mathbf{r}) & {}^{4}(i\mathbf{rr}\times\boldsymbol{\rho}) & {}^{4}(i\boldsymbol{\rho}\mathbf{r}\times\boldsymbol{\rho}) \end{array} \tag{4.1}$$

Cyclic versions of the doublet functions are formed through (2.11) and ${}^2P$ analogs of (2.11). Linear combinations of these and other functions from (4.1) also appear in the basis functions listed in Appendix A.

Certain groupings appear frequently, and have been given the following abbreviations:

$$(+) = [2r^2 - 3\rho^2/2 + i2(3)^{1/2}\mathbf{r}\cdot\boldsymbol{\rho}] \tag{4.2a}$$

$$(-) = [2r^2 - 3\rho^2/2 - i2(3)^{1/2}\mathbf{r}\cdot\boldsymbol{\rho}] \tag{4.2b}$$

$$[\ \ ] = 4\,{}^4(\mathbf{rr}) + 3\,{}^4(\boldsymbol{\rho\rho}) \tag{4.3a}$$

$$[+] = 4\,{}^4(\mathbf{rr}) - 3\,{}^4(\boldsymbol{\rho\rho}) + i2(3)^{1/2}\,{}^4(\mathbf{r}\boldsymbol{\rho}+\boldsymbol{\rho}\mathbf{r}) \tag{4.3b}$$

$$[-] = 4\,{}^4(\mathbf{rr}) - 3\,{}^4(\boldsymbol{\rho\rho}) - i2(3)^{1/2}\,{}^4(\mathbf{r}\boldsymbol{\rho}+\boldsymbol{\rho}\mathbf{r}) \tag{4.3c}$$

$$\mathbf{r}^+ = \mathbf{r} + i(3)^{1/2}\boldsymbol{\rho}/2 \tag{4.4a}$$

$$\mathbf{r}^- = \mathbf{r} - i(3)^{1/2}\boldsymbol{\rho}/2 \tag{4.4b}$$

A few of the functions are given in two forms, the second being the cyclic-notation form analogous to (2.17). For the most part, only the cyclic version is given, because of its compact character and its pertinence to the organization of these functions into families, as will be discussed in the next article in this set.

5. MATRIX FORMS OF WAVE EQUATIONS

Appendix B contains 36 rows of the rest-system wave equation (3.1), in the matrix form that uses the functions of Appendix A as basis functions. As indicated in (3.1), this is an inner-tier matrix applicable specifically to the upper-middle tier, with $M_\tau = +\frac{1}{2}$. However, a simple reversal of the sign of the left-hand side of each equation in Appendix B will make each equation there apply to the lower-middle tier, with $M_\tau = -\frac{1}{2}$. The appropriate specification of M_τ needs to be made in the τ-spin functions contained as factors in the basis functions of Appendix A.

It will be noted that the matrix rows in Appendix B are written in terms of the coefficients C_j instead of the functions ψ_j. The matrix elements were originally obtained through operations on the functions ψ_j. This gives a set of equations, the first of which is

$$H_r^{+1/2}\psi_1^{+1/2} = 2^{1/2}\psi_2^{+1/2} + 2^{1/2}\psi_3^{+1/2} \tag{5.1}$$

once the appropriate relative normalization has been incorporated into the functions ψ_j. This normalization ensures that the matrix is symmetrical.

The coefficients C_j satisfy a matrix equation whose matrix is the transpose of the matrix that is generated by operations such as (5.1). However, since the matrix has been made symmetrical through the choice of suitable normalization factors, the transposed matrix is unchanged, and the same matrix elements appear in the C_j equations of Appendix B as would appear in a listing of ψ_j equations such as (5.1).

The operator $H_r^{+1/2}$ in (2.22d) couples even-parity functions to odd-parity functions, and vice versa. Thus it is elementary to use some of the equations of Appendix B to eliminate the odd-parity coefficients (that is, the C_j's associated with ψ_j's which have odd parity) from the remaining equations. What is obtained is the set of equations listed in Appendix C. These are, in effect, the rows of the matrix version of the quadratic wave equation

$$m^2\Phi^{+1/2} = (H_r^{+1/2})^2\Phi^{+1/2} \tag{5.2}$$

that can be derived from (3.1).

That is, these are the even-parity rows of (5.2). As we will see later when this equation system is solved, we will not need to use the analogous odd-parity rows of (5.2).

6. SUMMARY

The expansion of the trilocal wave function has been started, with this initial expansion restricted to the rest system and to a time slice within which the two relative-time variables t_r and t_ρ are equal to zero.

The wave function lies in four tiers, specified by the ζ component of the total τ spin. For the upper-middle tier, where $M_\tau = +\frac{1}{2}$, the procedure for generating the expansion has been described. This involves repeated application of the appropriate Hamiltonian operator, given in (2.22d). Acting on one function, this operator generates a number of terms which include groupings recognized as functions already defined, and other terms which are grouped as tentative functions. Acting on the tentative functions, the operator generates previous functions and new functions. The associated matrix elements should form a symmetrical matrix, and this requires that normalization factors be adjusted, and sometimes that terms be regrouped until the correct groupings are found.

Forty-nine of the functions obtained in this way are listed here in Appendix A. The matrix elements linking them, in the matrix version of the linear wave equation, are included in Appendix B. The quadratic wave equation also has a matrix version, and Appendix C contains the matrix elements linking those functions in Appendix A which have even parity.

Further use of these functions and matrices will occur in later articles in this set.

7. ACKNOWLEDGMENTS

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APPENDIX A: 49 TRILOCAL BASIS FUNCTIONS

$$\psi_1 = N_0 h_0(\kappa\mathscr{R})[{}^b\Gamma\,{}^{2c}(1) - {}^c\Gamma\,{}^{2b}(1)]$$
$$= iN_0 2(3^{1/2}/3) h_0(\kappa\mathscr{R})[{}^-\Gamma\,{}^{2+}(1) - {}^+\Gamma\,{}^{2-}(1)]$$

$$\psi_2 = iN_0 2^{1/2}\kappa h_1(\kappa\mathscr{R})\,{}^s\Gamma[2\,{}^{2b}(\mathbf{r}) - {}^{2c}(\boldsymbol{\rho})]$$
$$= N_0 2(6^{1/2}/3)\kappa h_1(\kappa\mathscr{R})\,{}^s\Gamma[{}^{2+}(\mathbf{r}^-) - {}^{2-}(\mathbf{r}^+)]$$

$$\psi_3 = N_0(6^{1/2}/3)\kappa h_1(\kappa\mathscr{R})[{}^-\Gamma\,{}^4(\mathbf{r}^+) - {}^+\Gamma\,{}^4(\mathbf{r}^-)]$$

$$\psi_4 = N_0 2(6^{1/2}/3)\kappa h_1(\kappa\mathscr{R})[{}^+\Gamma\,{}^{2+}(\mathbf{r}^+) - {}^-\Gamma\,{}^{2-}(\mathbf{r}^-)]$$

$$\psi_5 = iN_0 2(6^{1/2}/3)\kappa^2 h_2(\kappa\mathscr{R})[{}^-\Gamma\,{}^{2-}(1)(+) - {}^+\Gamma\,{}^{2+}(1)(-)]$$

$$\psi_6 = iN_0 2(6^{1/2}/3)\kappa^2 h_2(\kappa\mathscr{R})\,{}^s\Gamma[{}^{2-}(1)(-) - {}^{2+}(1)(+)]$$

$$\psi_7 = N_0 4(2^{1/2})\kappa^2 h_2(\kappa\mathscr{R})[{}^{+}\Gamma\,{}^{2-}(i\mathbf{r}\times\boldsymbol{\rho}) + {}^{-}\Gamma\,{}^{2+}(i\mathbf{r}\times\boldsymbol{\rho})]$$

$$\psi_8 = N_0 4\kappa^2 h_2(\kappa\mathscr{R})\,{}^{s}\Gamma\,{}^{4}(i\mathbf{r}\times\boldsymbol{\rho})$$

$$\psi_9 = iN_0(15^{1/2}/5)\kappa^2 h_2(\kappa\mathscr{R})({}^{-}\Gamma[-] - {}^{+}\Gamma[+])$$

$$\psi_{10} = iN_0 8(3^{1/2})\kappa^3 h_3(\kappa\mathscr{R})[{}^{+}\Gamma\,{}^{4}(i\mathbf{r}^{-}\mathbf{r}\times\boldsymbol{\rho}) + {}^{-}\Gamma\,{}^{4}(i\mathbf{r}^{+}\mathbf{r}\times\boldsymbol{\rho})]$$

$$\psi_{11} = N_0 2(6^{1/2}/3)\kappa^3 h_3(\kappa\mathscr{R})\{{}^{+}\Gamma[{}^{4}(\mathbf{r}^{+})(-) - {}^{4}(\mathbf{r}^{-})\mathscr{R}^2/2] - {}^{-}\Gamma[{}^{4}(\mathbf{r}^{-})(+) - {}^{4}(\mathbf{r}^{+})\mathscr{R}^2/2]\}$$

$$\psi_{12} = N_0 2(3^{1/2}/3)\kappa^3 h_3(\kappa\mathscr{R})\,{}^{s}\Gamma[{}^{4}(\mathbf{r}^{+})(+) - {}^{4}(\mathbf{r}^{-})(-)]$$

$$\psi_{13} = N_0 4(6^{1/2}/3)\kappa^3 h_3(\kappa\mathscr{R})\{{}^{+}\Gamma[{}^{2+}(\mathbf{r}^{-})(+) - {}^{2+}(\mathbf{r}^{+})\mathscr{R}^2/2] - {}^{-}\Gamma[{}^{2-}(\mathbf{r}^{+})(-) - {}^{2-}(\mathbf{r}^{-})\mathscr{R}^2/2]\}$$

$$\psi_{14} = N_0 4(6^{1/2}/3)\kappa^3 h_3(\kappa\mathscr{R})\,{}^{s}\Gamma\{[{}^{2+}(\mathbf{r}^{+})(-) - {}^{2+}(\mathbf{r}^{-})\mathscr{R}^2/2] - [{}^{2-}(\mathbf{r}^{-})(+) - {}^{2-}(\mathbf{r}^{+})\mathscr{R}^2/2]\}$$

$$\psi_{15} = N_0 4(3^{1/2}/3)\kappa^3 h_3(\kappa\mathscr{R})[{}^{+}\Gamma\,{}^{2-}(\mathbf{r}^{+})(+) - {}^{-}\Gamma\,{}^{2+}(\mathbf{r}^{-})(-)]$$

$$\psi_{16} = N_0 4(3^{1/2}/3)\kappa^3 h_3(\kappa\mathscr{R})[{}^{+}\Gamma\,{}^{2-}(\mathbf{r}^{-})(-) - {}^{-}\Gamma\,{}^{2+}(\mathbf{r}^{+})(+)]$$

$$\psi_{17} = iN_0 2\kappa^4 h_4(\kappa\mathscr{R})\,{}^{s}\Gamma[{}^{2+}(1)(-)^2 - {}^{2-}(1)(+)^2]$$

$$\psi_{18} = iN_0 4\kappa^4 h_4(\kappa\mathscr{R})[{}^{-}\Gamma\,{}^{2+}(1) - {}^{+}\Gamma\,{}^{2-}(1)][(+)(-) - \mathscr{R}^4/2]$$

$$\psi_{19} = iN_0 2\kappa^4 h_4(\kappa\mathscr{R})[{}^{+}\Gamma\,{}^{2+}(1)(+)^2 - {}^{-}\Gamma\,{}^{2-}(1)(-)^2]$$

$$\psi_{20} = -N_0 2(6^{1/2})\kappa^4 h_4(\kappa\mathscr{R})\,{}^{4}(i\mathbf{r}\times\boldsymbol{\rho})[{}^{+}\Gamma(+) + {}^{-}\Gamma(-)]$$

$$\psi_{21} = -N_0 4(6^{1/2})\kappa^4 h_4(\kappa\mathscr{R})\,{}^{s}\Gamma[{}^{2+}(i\mathbf{r}\times\boldsymbol{\rho})(+) + {}^{2-}(i\mathbf{r}\times\boldsymbol{\rho})(-)]$$

$$\psi_{22} = N_0 4(6^{1/2})\kappa^4 h_4(\kappa\mathscr{R})[{}^{+}\Gamma\,{}^{2+}(i\mathbf{r}\times\boldsymbol{\rho})(-) + {}^{-}\Gamma\,{}^{2-}(i\mathbf{r}\times\boldsymbol{\rho})(+)]$$

$$\psi_{23} = iN_0 3^{1/2}\kappa^4 h_4(\kappa\mathscr{R})\,{}^{s}\Gamma[[+](-) - [-](+)]$$

$$\psi_{24} = iN_0 3(7^{1/2}/7)\kappa^4 h_4(\kappa\mathscr{R})[{}^{+}\Gamma[-](-) - {}^{-}\Gamma[+](+)]$$

$$\psi_{25} = iN_0(210^{1/2}/7)\kappa^4 h_4(\kappa\mathscr{R})\{{}^{-}\Gamma[[\quad](-) - [-]2\mathscr{R}^2/5] - {}^{+}\Gamma[[\quad](+) - [+]2\mathscr{R}^2/5]\}$$

$$\psi_{26} = N_0 2^{1/2}\kappa^5 h_5(\kappa\mathscr{R})[{}^{-}\Gamma\,{}^{4}(\mathbf{r}^{-})(-)^2 - {}^{+}\Gamma\,{}^{4}(\mathbf{r}^{+})(+)^2]$$

$$\psi_{27} = N_0 6^{1/2}\kappa^5 h_5(\kappa\mathscr{R})\,{}^{s}\Gamma\{{}^{4}(\mathbf{r}^{-})[(+)^2 + 2(-)\mathscr{R}^2/3] - {}^{4}(\mathbf{r}^{+})[(-)^2 + 2(+)\mathscr{R}^2/3]\}$$

$$\psi_{28} = N_0 2(3^{1/2})\kappa^5 h_5(\kappa\mathscr{R})\{[{}^{-}\Gamma\,{}^{4}(\mathbf{r}^{+}) - {}^{+}\Gamma\,{}^{4}(\mathbf{r}^{-})][(+)(-) - \mathscr{R}^4/3] - [{}^{-}\Gamma\,{}^{4}(\mathbf{r}^{-})(+) - {}^{+}\Gamma\,{}^{4}(\mathbf{r}^{+})(-)]\mathscr{R}^2/3\}$$

$$\psi_{29} = N_0 2(2^{1/2})\kappa^5 h_5(\kappa\mathscr{R})\,{}^{s}\Gamma[{}^{2-}(\mathbf{r}^{-})(-)^2 - {}^{2+}(\mathbf{r}^{+})(+)^2]$$

$$\psi_{30} = N_0 2(6^{1/2})\kappa^5 h_5(\kappa\mathscr{R})\{{}^{+}\Gamma[{}^{2-}(\mathbf{r}^{+}(-)^2 - {}^{2-}(\mathbf{r}^{-})2(-)\mathscr{R}^2/3] - {}^{-}\Gamma[{}^{2+}(\mathbf{r}^{-})(+)^2 - {}^{2+}(\mathbf{r}^{+})2(+)\mathscr{R}^2/3]\}$$

$$\psi_{31} = N_0 4(3^{1/2})\kappa^5 h_5(\kappa\mathscr{R})\, {}^s\Gamma\{[{}^{2-}(\mathbf{r}^+) - {}^{2+}(\mathbf{r}^-)][(+)(-) - \mathscr{R}^4/3] - [{}^{2-}(\mathbf{r}^-)(+) - {}^{2+}(\mathbf{r}^+)(-)]\mathscr{R}^2/3\}$$

$$\psi_{32} = N_0 2(6^{1/2})\kappa^5 h_5(\kappa\mathscr{R})\{{}^-\Gamma[{}^{2+}(\mathbf{r}^+)(-)^2 - {}^{2+}(\mathbf{r}^-)2(-)\mathscr{R}^2/3] - {}^+\Gamma[{}^{2-}(\mathbf{r}^-)(+)^2 - {}^{2-}(\mathbf{r}^+)2(+)\mathscr{R}^2/3]\}$$

$$\psi_{33} = N_0 2(2^{1/2})\kappa^5 h_5(\kappa\mathscr{R})[{}^+\Gamma\, {}^{2+}(\mathbf{r}^-)(-)^2 - {}^-\Gamma\, {}^{2-}(\mathbf{r}^+)(+)^2]$$

$$\psi_{34} = N_0 4(3^{1/2})\kappa^5 h_5(\kappa\mathscr{R})\{[{}^+\Gamma\, {}^{2+}(\mathbf{r}^+) - {}^-\Gamma\, {}^{2-}(\mathbf{r}^-)][(+)(-) - \mathscr{R}^4/3] - [{}^+\Gamma\, {}^{2+}(\mathbf{r}^-)(+) - {}^-\Gamma\, {}^{2-}(\mathbf{r}^+)(-)]\mathscr{R}^2/3\}$$

$$\psi_{35} = iN_0 24\kappa^5 h_5(\kappa\mathscr{R})\, {}^s\Gamma[{}^4(i\mathbf{r}^-\mathbf{r} \times \boldsymbol{\rho})(-) + {}^4(i\mathbf{r}^+\mathbf{r} \times \boldsymbol{\rho})(+)]$$

$$\psi_{36} = iN_0 12(6^{1/2})\kappa^5 h_5(\kappa\mathscr{R})\{{}^-\Gamma[{}^4(i\mathbf{r}^-\mathbf{r} \times \boldsymbol{\rho})(+) - {}^4(i\mathbf{r}^+\mathbf{r} \times \boldsymbol{\rho})\mathscr{R}^2/3] - {}^+\Gamma[{}^4(i\mathbf{r}^+\mathbf{r} \times \boldsymbol{\rho})(-) - {}^4(i\mathbf{r}^-\mathbf{r} \times \boldsymbol{\rho})\mathscr{R}^2/3]\}$$

$$\psi_{37} = iN_0 4(3^{1/2}/3)\kappa^6 h_6(\kappa\mathscr{R})[{}^-\Gamma\, {}^{2+}(1)(-)^3 - {}^+\Gamma\, {}^{2-}(1)(+)^3]$$

$$\psi_{38} = iN_0 4(3^{1/2})\kappa^6 h_6(\kappa\mathscr{R})[{}^-\Gamma\, {}^{2-}(1)(+) - {}^+\Gamma\, {}^{2+}(1)(-)][(+)(-) - 2\mathscr{R}^4/3]$$

$$\psi_{39} = iN_0 4(3^{1/2})\kappa^6 h_6(\kappa\mathscr{R})\, {}^s\Gamma[{}^{2-}(1)(-) - {}^{2+}(1)(+)][(+)(-) - 2\mathscr{R}^4/3]$$

$$\psi_{40} = iN_0 4(3^{1/2}/3)\kappa^6 h_6(\kappa\mathscr{R})[{}^-\Gamma\, {}^{2+}(1)(+)^3 - {}^+\Gamma\, {}^{2-}(1)(-)^3]$$

$$\psi_{41} = N_0 8(3^{1/2})\kappa^6 h_6(\kappa\mathscr{R})[{}^+\Gamma\, {}^{2+}(i\mathbf{r} \times \boldsymbol{\rho})(+)^2 + {}^-\Gamma\, {}^{2-}(i\mathbf{r} \times \boldsymbol{\rho})(-)^2]$$

$$\psi_{42} = N_0(-24)\kappa^6 h_6(\kappa\mathscr{R})[{}^+\Gamma\, {}^{2-}(i\mathbf{r} \times \boldsymbol{\rho}) + {}^-\Gamma\, {}^{2+}(i\mathbf{r} \times \boldsymbol{\rho})][(+)(-) - \mathscr{R}^4/3]$$

$$\psi_{43} = N_0 8(3^{1/2})\kappa^6 h_6(\kappa\mathscr{R})\, {}^s\Gamma[{}^{2+}(i\mathbf{r} \times \boldsymbol{\rho})(-)^2 + {}^{2-}(i\mathbf{r} \times \boldsymbol{\rho})(+)^2]$$

$$\psi_{44} = N_0(-12)(2^{1/2})\kappa^6 h_6(\kappa\mathscr{R})\, {}^s\Gamma\, {}^4(i\mathbf{r} \times \boldsymbol{\rho})[(+)(-) - \mathscr{R}^4/3]$$

$$\psi_{45} = N_0 4(3^{1/2})\kappa^6 h_6(\kappa\mathscr{R})\, {}^4(i\mathbf{r} \times \boldsymbol{\rho})[{}^+\Gamma(-)^2 + {}^-\Gamma(+)^2]$$

$$\psi_{46} = iN_0 6(138^{1/2}/23)\kappa^6 h_6(\kappa\mathscr{R})\{[{}^-\Gamma[-] - {}^+\Gamma[+]][(+)(-) - \mathscr{R}^4/3] - [{}^-\Gamma[+](-)^2 - {}^+\Gamma[-](+)^2]\mathscr{R}^2/2\}$$

$$\psi_{47} = iN_0(2^{1/2})\kappa^6 h_6(\kappa\mathscr{R})\, {}^s\Gamma[[+](+)^2 - [-](-)^2]$$

$$\psi_{48} = iN_0 2(21^{1/2}/3)\kappa^6 h_6(\kappa\mathscr{R})\{{}^-\Gamma[[\quad](+)^2 - [+]4(+)\mathscr{R}^2/7] - {}^+\Gamma[[\quad](-)^2 - [-]4(-)\mathscr{R}^2/7]\}$$

$$\psi_{49} = iN_0 7(966^{1/2}/69)\kappa^6 h_6(\kappa\mathscr{R})\{[{}^+\Gamma[+] - {}^-\Gamma[-]][(+)(-) - \mathscr{R}^4/49] + {}^+\Gamma[[-]8(+)^2/7 - [\quad]92(+)\mathscr{R}^2/49] - {}^-\Gamma[[+]8(-)^2/7 - [\quad]92(-)\mathscr{R}^2/49]\}$$

APPENDIX B: 36 ROWS OF THE INNER-TIER LINEAR WAVE EQUATION

$$mC_1 = 2^{1/2}C_2 + 2^{1/2}C_3$$

$$mC_2 = 2^{1/2}C_1 + (1/3)C_5 - (1/3)C_6 + (1/3)C_7 + (2^{1/2}/6)C_8 + (10^{1/2}/3)C_9$$

$$mC_3 = 2^{1/2}C_1 - (1/3)C_5 - (2/3)C_6 + (1/6)C_7 + 5(2^{1/2}/6)C_8 + (10^{1/2}/6)C_9$$

$$mC_4 = (2/3)C_5 + (1/3)C_6 + (2/3)C_7 + (2^{1/2}/3)C_8 + (10^{1/2}/6)C_9$$

$$mC_5 = (1/3)C_2 - (1/3)C_3 + (2/3)C_4 - (1/3)C_{11} - 2(2^{1/2}/3)C_{12} - (2/3)C_{13} - (1/3)C_{14} + (2^{1/2}/3)C_{15} - (2^{1/2}/3)C_{16}$$

$$mC_6 = -(1/3)C_2 - (2/3)C_3 + (1/3)C_4 - (2/3)C_{11} - (2^{1/2}/3)C_{12} - (1/3)C_{13} + (1/3)C_{14} + 2(2^{1/2}/3)C_{15} + (2^{1/2}/3)C_{16}$$

$$mC_7 = (1/3)C_2 + (1/6)C_3 + (2/3)C_4 + (6^{1/2}/6)C_{10} - (1/6)C_{11} + (2/3)C_{13} + (1/3)C_{14}$$

$$mC_8 = (2^{1/2}/6)C_2 + 5(2^{1/2}/6)C_3 + (2^{1/2}/3)C_4 - (3^{1/2}/3)C_{10} - 5(2^{1/2}/6)C_{11} + (2^{1/2}/3)C_{13} + (2^{1/2}/6)C_{14}$$

$$mC_9 = (10^{1/2}/3)C_2 + (10^{1/2}/6)C_3 + (10^{1/2}/6)C_4 - (15^{1/2}/5)C_{10} - (10^{1/2}/30)C_{11} - 2(5^{1/2}/15)C_{12} + (10^{1/2}/30)C_{13} + (10^{1/2}/15)C_{14} - 2(5^{1/2}/15)C_{15} - 4(5^{1/2}/15)C_{16}$$

$$mC_{10} = (6^{1/2}/6)C_7 - (3^{1/2}/3)C_8 - (15^{1/2}/5)C_9 - (2^{1/2}/6)C_{20} + (2^{1/2}/3)C_{21} + (2^{1/2}/6)C_{22} - (3^{1/2}/3)C_{23} - (210^{1/2}/30)C_{25}$$

$$mC_{11} = -(1/3)C_5 - (2/3)C_6 - (1/6)C_7 - 5(2^{1/2}/6)C_8 - (10^{1/2}/30)C_9 + (6^{1/2}/3)C_{18} - 5(3^{1/2}/9)C_{20} - 2(3^{1/2}/9)C_{21} - (3^{1/2}/9)C_{22} - (2^{1/2}/3)C_{23} - (35^{1/2}/15)C_{25}$$

$$mC_{12} = -2(2^{1/2}/3)C_5 - (2^{1/2}/3)C_6 - 2(5^{1/2}/15)C_9 + 2(3^{1/2}/9)C_{17} + 4(3^{1/2}/9)C_{19} - 5(6^{1/2}/18)C_{20} + (6^{1/2}/18)C_{21} + (6^{1/2}/9)C_{22} + (21^{1/2}/9)C_{24} + (70^{1/2}/30)C_{25}$$

$$mC_{13} = -(2/3)C_5 - (1/3)C_6 + (2/3)C_7 + (2^{1/2}/3)C_8 + (10^{1/2}/30)C_9 - (3^{1/2}/9)C_{20} + 2(3^{1/2}/9)C_{21} + 4(3^{1/2}/9)C_{22} - 2(2^{1/2}/3)C_{23} + (35^{1/2}/15)C_{25}$$

$$mC_{14} = -(1/3)C_5 + (1/3)C_6 + (1/3)C_7 + (2^{1/2}/6)C_8 + (10^{1/2}/15)C_9 - (6^{1/2}/3)C_{18} - 2(3^{1/2}/9)C_{20} - 2(3^{1/2}/9)C_{21} + 2(3^{1/2}/9)C_{22} - (2^{1/2}/3)C_{23} + 2(35^{1/2}/15)C_{25}$$

$$mC_{15} = (2^{1/2}/3)C_5 + 2(2^{1/2}/3)C_6 - 2(5^{1/2}/15)C_9 + 2(3^{1/2}/9)C_{17} - 2(3^{1/2}/9)C_{19} + (6^{1/2}/18)C_{20} + 2(6^{1/2}/9)C_{21} + (6^{1/2}/9)C_{22} - 2(21^{1/2}/9)C_{24} + (70^{1/2}/30)C_{25}$$

$$mC_{16} = -(2^{1/2}/3)C_5 + (2^{1/2}/3)C_6 - 4(5^{1/2}/15)C_9 + 4(3^{1/2}/9)C_{17} + 2(3^{1/2}/9)C_{19} + (6^{1/2}/9)C_{20} + (6^{1/2}/9)C_{21} - (6^{1/2}/9)C_{22} - (21^{1/2}/9)C_{24} + (70^{1/2}/15)C_{25}$$

$$mC_{17} = 2(3^{1/2}/9)C_{12} + 2(3^{1/2}/9)C_{15} + 4(3^{1/2}/9)C_{16} - 2(2^{1/2}/3)C_{26} - (6^{1/2}/9)C_{27} + (2^{1/2}/3)C_{29} - 2(6^{1/2}/9)C_{30} + (6^{1/2}/9)C_{32} + (2^{1/2}/3)C_{33}$$

$$mC_{18} = (6^{1/2}/3)C_{11} - (6^{1/2}/3)C_{14} + 2(3^{1/2}/3)C_{28} - 2(3^{1/2}/3)C_{31}$$

$$mC_{19} = 4(3^{1/2}/9)C_{12} - 2(3^{1/2}/9)C_{15} + 2(3^{1/2}/9)C_{16} - (2^{1/2}/3)C_{26} - 2(6^{1/2}/9)C_{27} - (2^{1/2}/3)C_{29} - (6^{1/2}/9)C_{30} - (6^{1/2}/9)C_{32} + 2(2^{1/2}/3)C_{33}$$

$$mC_{20} = -(2^{1/2}/6)C_{10} - 5(3^{1/2}/9)C_{11} - 5(6^{1/2}/18)C_{12} - (3^{1/2}/9)C_{13} - 2(3^{1/2}/9)C_{14} + (6^{1/2}/18)C_{15} + (6^{1/2}/9)C_{16} - 5(3^{1/2}/9)C_{27} + 5(6^{1/2}/18)C_{28} + 2(3^{1/2}/9)C_{30} + (6^{1/2}/9)C_{31} - (3^{1/2}/9)C_{32} - (6^{1/2}/18)C_{34} + (6^{1/2}/6)C_{35} + (1/3)C_{36}$$

$$mC_{21} = (2^{1/2}/3)C_{10} - 2(3^{1/2}/9)C_{11} + (6^{1/2}/18)C_{12} + 2(3^{1/2}/9)C_{13} - 2(3^{1/2}/9)C_{14} + 2(6^{1/2}/9)C_{15} + (6^{1/2}/9)C_{16} + (3^{1/2}/9)C_{27} + (6^{1/2}/9)C_{28} + 2(3^{1/2}/9)C_{30} + (6^{1/2}/9)C_{31} - 4(3^{1/2}/9)C_{32} + (6^{1/2}/9)C_{34} + (6^{1/2}/6)C_{35} - (2/3)C_{36}$$

$$mC_{22} = (2^{1/2}/6)C_{10} - (3^{1/2}/9)C_{11} + (6^{1/2}/9)C_{12} + 4(3^{1/2}/9)C_{13} + 2(3^{1/2}/9)C_{14} + (6^{1/2}/9)C_{15} - (6^{1/2}/9)C_{16} + 2(3^{1/2}/9)C_{27} + (6^{1/2}/18)C_{28} - 2(3^{1/2}/9)C_{30} - (6^{1/2}/9)C_{31} - 2(3^{1/2}/9)C_{32} + 2(6^{1/2}/9)C_{34} + (6^{1/2}/3)C_{35} - (1/3)C_{36}$$

$$mC_{23} = -(3^{1/2}/3)C_{10} - (2^{1/2}/3)C_{11} - 2(2^{1/2}/3)C_{13} - (2^{1/2}/3)C_{14} + (1/3)C_{28} + (1/3)C_{31} - (2/3)C_{34} + (6^{1/2}/3)C_{36}$$

$$mC_{24} = (21^{1/2}/9)C_{12} - 2(21^{1/2}/9)C_{15} - (21^{1/2}/9)C_{16} + 2(14^{1/2}/21)C_{26} + (42^{1/2}/63)C_{27} - 4(14^{1/2}/21)C_{29} - (42^{1/2}/63)C_{30} + 2(42^{1/2}/63)C_{32} + 2(14^{1/2}/21)C_{33} - (21^{1/2}/7)C_{35}$$

$$mC_{25} = -(210^{1/2}/30)C_{10} - (35^{1/2}/15)C_{11} + (70^{1/2}/30)C_{12} + (35^{1/2}/15)C_{13} + 2(35^{1/2}/15)C_{14} + (70^{1/2}/30)C_{15} + (70^{1/2}/15)C_{16} - (35^{1/2}/21)C_{27} - (70^{1/2}/42)C_{28} - 2(35^{1/2}/21)C_{30} + (70^{1/2}/21)C_{31} + (35^{1/2}/21)C_{32} - (70^{1/2}/42)C_{34} + (70^{1/2}/14)C_{35} - (105^{1/2}/21)C_{36}$$

$$mC_{26} = -2(2^{1/2}/3)C_{17} - (2^{1/2}/3)C_{19} + 2(14^{1/2}/21)C_{24} - (6^{1/2}/6)C_{37} - (6^{1/2}/3)C_{40} + (2^{1/2}/12)C_{41} - (2^{1/2}/6)C_{43} - 5(2^{1/2}/12)C_{45} + (1/2)C_{47} + (42^{1/2}/28)C_{48}$$

$$mC_{27} = -(6^{1/2}/9)C_{17} - 2(6^{1/2}/9)C_{19} - 5(3^{1/2}/9)C_{20} + (3^{1/2}/9)C_{21} + 2(3^{1/2}/9)C_{22} + (42^{1/2}/63)C_{24} - (35^{1/2}/21)C_{25} + (2^{1/2}/3)C_{38} + (2^{1/2}/6)C_{39} - (6^{1/2}/6)C_{41} + (6^{1/2}/12)C_{43} - 5(6^{1/2}/12)C_{45} - 7(23^{1/2}/138)C_{46} - 3(14^{1/2}/28)C_{48} - 3(161^{1/2}/161)C_{49}$$

$$mC_{28} = 2(3^{1/2}/3)C_{18} + 5(6^{1/2}/18)C_{20} + (6^{1/2}/9)C_{21} + (6^{1/2}/18)C_{22} + (1/3)C_{23} - (70^{1/2}/42)C_{25} - (1/3)C_{38} - (2/3)C_{39} - (1/6)C_{42} - 5(2^{1/2}/6)C_{44} + 4(46^{1/2}/69)C_{46} - 3(322^{1/2}/322)C_{49}$$

$$mC_{29} = (2^{1/2}/3)C_{17} - (2^{1/2}/3)C_{19} - 4(14^{1/2}/21)C_{24} + (6^{1/2}/3)C_{37} + (6^{1/2}/6)C_{40} - (2^{1/2}/6)C_{41} - (2^{1/2}/6)C_{43} - (2^{1/2}/6)C_{45} + (1/2)C_{47} - (42^{1/2}/14)C_{48}$$

$$
\begin{aligned}
mC_{30} = {} & -2(6^{1/2}/9)C_{17} - (6^{1/2}/9)C_{19} + 2(3^{1/2}/9)C_{20} + 2(3^{1/2}/9)C_{21} \\
& - 2(3^{1/2}/9)C_{22} - (42^{1/2}/63)C_{24} - 2(35^{1/2}/21)C_{25} + (2^{1/2}/6)C_{38} \\
& - (2^{1/2}/6)C_{39} + (6^{1/2}/6)C_{41} - (6^{1/2}/3)C_{43} - (6^{1/2}/12)C_{45} \\
& - 7(23^{1/2}/69)C_{46} + 3(14^{1/2}/28)C_{48} - 6(161^{1/2}/161)C_{49}
\end{aligned}
$$

$$
\begin{aligned}
mC_{31} = {} & -2(3^{1/2}/3)C_{18} + (6^{1/2}/9)C_{20} + (6^{1/2}/9)C_{21} - (6^{1/2}/9)C_{22} + (1/3)C_{23} \\
& + (70^{1/2}/21)C_{25} - (1/3)C_{38} + (1/3)C_{39} + (1/3)C_{42} + (2^{1/2}/6)C_{44} \\
& - 8(46^{1/2}/69)C_{46} \quad 3(322^{1/2}/161)C_{49}
\end{aligned}
$$

$$
\begin{aligned}
mC_{32} = {} & (6^{1/2}/9)C_{17} - (6^{1/2}/9)C_{19} - (3^{1/2}/9)C_{20} - 4(3^{1/2}/9)C_{21} \\
& - 2(3^{1/2}/9)C_{22} + 2(42^{1/2}/63)C_{24} + (35^{1/2}/21)C_{25} + (2^{1/2}/6)C_{38} \\
& + (2^{1/2}/3)C_{39} + (6^{1/2}/6)C_{41} + (6^{1/2}/6)C_{43} + (6^{1/2}/6)C_{45} \\
& + 7(23^{1/2}/138)C_{46} - 3(14^{1/2}/14)C_{48} + 3(161^{1/2}/161)C_{49}
\end{aligned}
$$

$$
\begin{aligned}
mC_{33} = {} & (2^{1/2}/3)C_{17} + 2(2^{1/2}/3)C_{19} + 2(14^{1/2}/21)C_{24} \\
& - (6^{1/2}/6)C_{37} + (6^{1/2}/6)C_{40} + (2^{1/2}/3)C_{41} \\
& - (2^{1/2}/6)C_{43} + (2^{1/2}/12)C_{45} - C_{47} + (42^{1/2}/28)C_{48}
\end{aligned}
$$

$$
\begin{aligned}
mC_{34} = {} & -(6^{1/2}/18)C_{20} + (6^{1/2}/9)C_{21} + 2(6^{1/2}/9)C_{22} \\
& - (2/3)C_{23} - (70^{1/2}/42)C_{25} + (2/3)C_{38} + (1/3)C_{39} \\
& - (2/3)C_{42} - (2^{1/2}/3)C_{44} + 4(46^{1/2}/69)C_{46} - 3(322^{1/2}/322)C_{49}
\end{aligned}
$$

$$
\begin{aligned}
mC_{35} = {} & (6^{1/2}/6)C_{20} + (6^{1/2}/6)C_{21} + (6^{1/2}/3)C_{22} - (21^{1/2}/7)C_{24} \\
& + (70^{1/2}/14)C_{25} - (3^{1/2}/3)C_{41} + (3^{1/2}/6)C_{43} + (3^{1/2}/6)C_{45} \\
& + 3(46^{1/2}/46)C_{46} + 3(7^{1/2}/14)C_{48} - 6(322^{1/2}/161)C_{49}
\end{aligned}
$$

$$
\begin{aligned}
mC_{36} = {} & (1/3)C_{20} - (2/3)C_{21} - (1/3)C_{22} + (6^{1/2}/3)C_{23} - (105^{1/2}/21)C_{25} \\
& + (6^{1/2}/12)C_{42} - (3^{1/2}/6)C_{44} + (69^{1/2}/23)C_{46} + 15(483^{1/2}/322)C_{49}
\end{aligned}
$$

APPENDIX C: 15 ROWS OF THE INNER-TIER QUADRATIC WAVE EQUATION

$$
0 = (4 - m^2)C_1 - 2^{1/2}C_6 + (2^{1/2}/2)C_7 + 2C_8 + 5^{1/2}C_9
$$

$$
\begin{aligned}
0 = {} & [(8/3) - m^2]C_5 + (4/3)C_6 + 4(10^{1/2}/15)C_9 - 2(6^{1/2}/9)C_{17} \\
& - 4(6^{1/2}/9)C_{19} + 2(3^{1/2}/3)C_{20} - (3^{1/2}/3)C_{22} + 2(2^{1/2}/3)C_{23} \\
& - (42^{1/2}/9)C_{24} - 2(35^{1/2}/15)C_{25}
\end{aligned}
$$

$$
\begin{aligned}
0 = {} & -2^{1/2}C_1 + (4/3)C_5 + [(8/3) - m^2]C_6 - 4(10^{1/2}/15)C_9 \\
& + 2(6^{1/2}/9)C_{17} - (6^{1/2}/3)C_{18} - 2(6^{1/2}/9)C_{19} + 2(3^{1/2}/3)C_{20} \\
& + (3^{1/2}/3)C_{21} + (2^{1/2}/3)C_{23} - 2(42^{1/2}/9)C_{24} + 2(35^{1/2}/15)C_{25}
\end{aligned}
$$

$$
\begin{aligned}
0 = {} & (2^{1/2}/2)C_1 + [(4/3) - m^2]C_7 + 2(2^{1/2}/3)C_8 + (10^{1/2}/5)C_9 \\
& - (6^{1/2}/6)C_{18} - (3^{1/2}/9)C_{20} + 2(3^{1/2}/9)C_{21} \\
& + 4(3^{1/2}/9)C_{22} - 2(2^{1/2}/3)C_{23} + (35^{1/2}/15)C_{25}
\end{aligned}
$$

$$\begin{aligned}0 = {} & 2C_1 + 2(2^{1/2}/3)C_7 + [(11/3) - m^2]C_8 + 4(5^{1/2}/5)C_9 \\ & - 2(3^{1/2}/3)C_{18} + 4(6^{1/2}/9)C_{20} + (6^{1/2}/9)C_{21} \\ & + 2(6^{1/2}/9)C_{22} + (1/3)C_{23} + 2(70^{1/2}/15)C_{25}\end{aligned}$$

$$\begin{aligned}0 = {} & 5^{1/2}C_1 + 4)10^{1/2}/15)C_5 - 4(10^{1/2}/15)C_6 + (10^{1/2}/5)C_7 + 4(5^{1/2}/5)C_8 \\ & + [(43/15) - m^2]C_9 - 8(15^{1/2}/45)C_{17} - (15^{1/2}/15)C_{18} - 4(15^{1/2}/45)C_{19} \\ & + (30^{1/2}/30)C_{20} - 2(30^{1/2}/15)C_{21} + 2(5^{1/2}/15)C_{23} \\ & + 2(105^{1/2}/45)C_{24} + (14^{1/2}/30)C_{25}\end{aligned}$$

$$\begin{aligned}0 = {} & -2(6^{1/2}/9)C_5 + 2(6^{1/2}/9)C_6 - 8(15^{1/2}/45)C_9 + [(8/3) - m^2]C_{17} \\ & + (4/3)C_{19} - 8(7^{1/2}/21)C_{24} + 8(210^{1/2}/105)C_{25} + (3^{1/2}/3)C_{37} \\ & - (3^{1/2}/9)C_{38} + (3^{1/2}/9)C_{39} + 2(3^{1/2}/3)C_{40} + (1/2)C_{43} + C_{45} \\ & + 7(138^{1/2}/207)C_{46} - (2^{1/2}/2)C_{47} - (21^{1/2}/7)C_{48} + 2(966^{1/2}/161)C_{49}\end{aligned}$$

$$\begin{aligned}0 = {} & -(6^{1/2}/3)C_6 - (6^{1/2}/6)C_7 - 2(3^{1/2}/3)C_8 - (15^{1/2}/15)C_9 + (4 - m^2)C_{18} \\ & - 4(210^{1/2}/35)C_{25} - 2(3^{1/2}/3)C_{39} - (3^{1/2}/3)C_{42} - 2(6^{1/2}/3)C_{44} \\ & + 8(138^{1/2}/69)C_{46} - 3(966^{1/2}/161)C_{49}\end{aligned}$$

$$\begin{aligned}0 = {} & -4(6^{1/2}/9)C_5 - 2(6^{1/2}/9)C_6 - 4(15^{1/2}/45)C_9 + (4/3)C_{17} \\ & + [(8/3) - m^2]C_{19} + 8(7^{1/2}/21)C_{24} + 4(210^{1/2}/105)C_{25} - (3^{1/2}/3)C_{37} \\ & - 2(3^{1/2}/9)C_{38} - (3^{1/2}/9)C_{39} + (3^{1/2}/3)C_{40} + (1/2)C_{41} + C_{45} \\ & + 7(138^{1/2}/414)C_{46} - 2^{1/2}C_{47} + (21^{1/2}/7)C_{48} + (966^{1/2}/161)C_{49}\end{aligned}$$

$$\begin{aligned}0 = {} & 2(3^{1/2}/3)C_5 + 2(3^{1/2}/3)C_6 - (3^{1/2}/9)C_7 + 4(6^{1/2}/9)C_8 + (30^{1/2}/30)C_9 \\ & + [(11/3) - m^2]C_{20} + (2/3)C_{21} - (2/3)C_{22} + 2(6^{1/2}/3)C_{23} \\ & - 2(14^{1/2}/7)C_{24} + 2(105^{1/2}/105)C_{25} - (6^{1/2}/3)C_{38} - (6^{1/2}/3)C_{39} \\ & + (2^{1/2}/6)C_{41} + (6^{1/2}/18)C_{42} - (2^{1/2}/3)C_{43} - 4(3^{1/2}/9)C_{44} \\ & + 2(2^{1/2}/3)C_{45} + 5(69^{1/2}/138)C_{46} + (42^{1/2}/7)C_{48} + (483^{1/2}/322)C_{49}\end{aligned}$$

$$\begin{aligned}0 = {} & (3^{1/2}/3)C_6 + 2(3^{1/2}/9)C_7 + (6^{1/2}/9)C_8 - 2(30^{1/2}/15)C_9 + (2/3)C_{20} \\ & + [(8/3) - m^2]C_{21} + (4/3)C_{22} - (6^{1/2}/3)C_{23} - 2(14^{1/2}/7)C_{24} \\ & + 2(105^{1/2}/105)C_{25} - (6^{1/2}/6)C_{39} - (2^{1/2}/3)C_{41} - (6^{1/2}/9)C_{42} \\ & - (2^{1/2}/3)C_{43} - (3^{1/2}/9)C_{44} - (2^{1/2}/3)C_{45} - 4(69^{1/2}/69)C_{46} \\ & + (42^{1/2}/7)C_{48} - 10(483^{1/2}/161)C_{49}\end{aligned}$$

$$\begin{aligned}0 = {} & -(3^{1/2}/3)C_5 + 4(3^{1/2}/9)C_7 + 2(6^{1/2}/9)C_8 - (2/3)C_{20} + (4/3)C_{21} \\ & + [(8/3) - m^2]C_{22} - 2(6^{1/2}/3)C_{23} - (14^{1/2}/7)C_{24} + 2(105^{1/2}/21)C_{25} \\ & + (6^{1/2}/6)C_{38} - 2(2^{1/2}/3)C_{41} - 2(6^{1/2}/9)C_{42} + (2^{1/2}/3)C_{43} \\ & - 2(3^{1/2}/9)C_{44} - (2^{1/2}/6)C_{45} + 2(69^{1/2}/23)C_{46} \\ & + (42^{1/2}/14)C_{48} - 8(483^{1/2}/161)C_{49}\end{aligned}$$

$$\begin{aligned}0 = {} & 2(2^{1/2}/3)C_5 + (2^{1/2}/3)C_6 - 2(2^{1/2}/3)C_7 + (1/3)C_8 + 2(5^{1/2}/15)C_9 \\ & + 2(6^{1/2}/3)C_{20} - (6^{1/2}/3)C_{21} - 2(6^{1/2}/3)C_{22} + (3 - m^2)C_{23} \\ & - 2(70^{1/2}/35)C_{25} - (2/3)C_{38} - (1/3)C_{39} + (2/3)C_{42} - (2^{1/2}/6)C_{44} \\ & - (46^{1/2}/69)C_{46} + 9(322^{1/2}/161)C_{49}\end{aligned}$$

$$\begin{aligned}0 = & -(42^{1/2}/9)C_5 - 2(42^{1/2}/9)C_6 + 2(105^{1/2}/45)C_9 - 8(7^{1/2}/21)C_{17} \\ & + 8(7^{1/2}/21)C_{19} - 2(14^{1/2}/7)C_{20} - 2(14^{1/2}/7)C_{21} - (14^{1/2}/7)C_{22} \\ & + [(59/21) - m^2]C_{24} - 2(30^{1/2}/15)C_{25} - 4(21^{1/2}/21)C_{37} + (21^{1/2}/63)C_{38} \\ & + 2(21^{1/2}/63)C_{39} - 2(21^{1/2}/21)C_{40} + 2(7^{1/2}/7)C_{41} - (7^{1/2}/14)C_{45} \\ & - 10(966^{1/2}/1449)C_{46} - (14^{1/2}/7)C_{47} - (3^{1/2}/14)C_{48} + (138^{1/2}/23)C_{49}\end{aligned}$$

$$\begin{aligned}0 = & -2(35^{1/2}/15)C_5 + 2(35^{1/2}/15)C_6 + (35^{1/2}/15)C_7 + 2(70^{1/2}/15)C_8 \\ & + (14^{1/2}/30)C_9 + 8(210^{1/2}/105)C_{17} - 4(210^{1/2}/35)C_{18} \\ & + 4(210^{1/2}/105)C_{19} + 2(105^{1/2}/105)C_{20} + 2(105^{1/2}/105)C_{21} \\ & + 2(105^{1/2}/21)C_{22} - 2(70^{1/2}/35)C_{23} - 2(30^{1/2}/15)C_{24} \\ & + [(103/35) - m^2]C_{25} - (70^{1/2}/21)C_{38} + (70^{1/2}/21)C_{39} \\ & - (210^{1/2}/42)C_{41} + (70^{1/2}/42)C_{42} + (210^{1/2}/21)C_{43} + 2(35^{1/2}/21)C_{44} \\ & + (210^{1/2}/21)C_{45} + (805^{1/2}/966)C_{46} - 9(115^{1/2}/322)C_{49}\end{aligned}$$

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